Existence and structure of past asymptotically simple solutions of Einstein's field equations with positive cosmological constant

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Abstract. *The initial value problem for Einstein's field equations with positive cosmological constant is analysed where data are prescribed at past conformal infinity. It is found that the data on past conformal infinity are given, up to arbitrary conformal rescalings, by a freely specifyble Riemannian metric and a trace-free, symmetric tensorfield of valence two, which satisfies a divergence equation. For each initial data set exists a unique (semi-global) past asymptotically simple solution of Einstein's equations. The case is discussed where in such a space-time exists a Killing vector field with a time-like trajectory which approaches a point p on conformal infinity. It is shown that in a neighbourhood of the trajectory near p the space-time is conformally flat.*

1. INTRODUCTION

In this paper will be shown the existence of a large class of solutions of Einstein's field equations (2.1) with positive cosmological constant Λ, which are asymptotically simple in the past. These solutions are semiglobal in the sense that all null geodesics are past complete. The existence proof is based on the properties of the regular conformal field equations [7, 8, 9]. They allow to characterize the solutions by their Cauchy data on past conformal infinity $\mathscr{I}^-$, which can be thought of as the set of past endopoints of the null geodesics. In

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the conformally extended space-time $\mathscr{I}^-$ forms a smooth space-like hypersurface. It turns out that on past conformal infinity the constraints become particularly easy to solve and one can give a simple and complete description of all initial data sets (based on conformal infinity) arising from solutions of Einstein's equations which are asymptotically simple (Lemma (3.1)). Two facts are remarkable here. There is no analogue of the Hamiltonian constraint. Consequently the (orientable) manifold $\mathscr{I}^-$ and the Riemannian metric implied on it by the conformal field are completely arbitrary (which appears to have been noticed by Starobinsky before [2]). Nevertheless, since for the physical field only the conformal equivalence class of the conformal initial data on $\mathscr{I}^-$ is relevant, the freedom to prescribe data on $\mathscr{I}^-$ is essentially the same as the freedom to describe data on a space-like hypersurface in the standard Cauchy problem for Einstein's field equations. This suggests that being weakly asymptotically simple, i.e. possessing «patches of a smooth conformal infinity», is a rather general feature of solutions of Einstein's field equation with $\Lambda > 0$.

The problem of showing existence of solutions of Einstein's equations with $\Lambda > 0$ for data given on past conformal infinity is similar to the «pure radiation problem» [10] where data for Einstein's equations with $\Lambda = 0$ are prescribed on a «regular cone with complete generators» representing past conformal infinity. However, the former case is much easier to deal with since here the initial surface is smooth and space-like. Thus the existence of solutions of the symmetric hyperbolic propagation equations implied by the regular conformal field equations can be inferred from well-known theorems [8, 16]. This immediately implies the existence of solutions of Einstein's field equations (2.1) which approach the data given on past conformal infinity (Theorem (3.2)).

In the last chapter a consequence of the existence of certain symmetries in the type of space-time is discussed, whose existence has been established before. It is shown (Theorem (4.1)) if a solution of Einstein's equations (2.1) ($\Lambda > 0$) which is weakly asymptotically simple admits a Killing vector field with a time--like integral curve which approaches a point p on $\mathscr{I}^+$, then the metric implied on a certain neighbourhood of p in $J^-(p)$, the casual past of p in the conformally extended space-time, is conformally flat. This fact, which has been conjectured and discussed (partly under strong conditions on the space-time) in [1, 2, 3, 13] is established here by a straightforward application of the techniques used in chapters 2, 3 and in [10].

The results presented here allow generalizations. In particular Lemma (3.1), Theorem (3.2), and Theorem (4.1) can be extended to the case of the coupled Einstein-Yang-Mills equations. Furthermore it may be pointed out that the regular conformal field equations may be used in a similar way to investigate the existence of asymptotically simple space-times satisfying Einstein's equations

(2.1) with $\Lambda < 0$. In that case one would have to discuss a mixed problem for the regular conformal field equations where data are given on a space-like hypersurface and on the time-like surface representing conformal infinity [20].

2. THE CONFORMAL STRUCTURE OF THE EQUATIONS AND OF THE FIELDS

In the next chapter will be shown the existence of solutions of Einstein's field equations with positive cosmological constant

$$\text{Ric}\,[\tilde{g}] = \Lambda \tilde{g} \tag{2.1}$$

for the «physical metric» $\tilde{g}$. In the existence theorem will be stated weak differentiability conditions on the data and the fields. In this chapter, however, and in chapter 4 will be assumed for convenience that all fields are of class C^∞.

The existence proof will be based on the properties of the «regular conformal field equations» [7, 8, 9]. These equations, which are derived from Einstein's equations by exploiting the «conformal structure» of the latter, constitute a slight generalization of (2.1) since they are regular and make sense in regions where Einstein's equations are no longer defined. If Ω is a positive function, called the «conformal factor» in the following, then one can define a «non-physical metric»

$$g_{\mu\nu} = \Omega^2 \tilde{g}_{\mu\nu} \tag{2.2}$$

and (2.1) may be expressed equivalently by the «conformal field equations»

$$\text{Ric}\,[\Omega^{-2} g] = \Lambda\, \Omega^{-2} g. \tag{2.3}$$

Denote by ∇ the covariant Levi-Civita derivate operator defined by $g_{\mu\nu}$ and let $e_k = e^\mu_k \dfrac{\partial}{\partial x^\mu}$ (where $i, k, l, \ldots = 0, 1, \ldots, 4$; $\mu, \nu, \lambda, \ldots = 0, 1, \ldots, 4$ and the summation convention is assumed) be an orthonormal frame for g

$$g(e_k, e_j) = g_{kj} = \text{diag}\,(-1, 1, 1, 1) \tag{2.4}$$

given with respect to some coordinate system x^μ. In the following all equations will be written in a frame formalism and expressed with respect to the frame e_k. The connection coefficients γ^j_{ik} are then defined by the directional covariant derivatives of the vector fields e_k

$$\gamma^j_{ik}\, e_j = \nabla_i\, e_k = \nabla_{e_i} e_k\,,\ \gamma^j_{ik}\, g_{jl} + \gamma^j_{il} g_{jk} = 0. \tag{2.5}$$

Besides Ω, e^μ_k, γ^j_{ik} the fields which are of interest in the following are:

– the differential $\Sigma_k = \Omega_{,\mu} e^\mu_k$ of Ω and $s = \frac{1}{4} \nabla_\mu \nabla^\mu \Omega$,

– the Ricci scalar R, the traceless part $2\, s_{ij} = R_{ij} - \frac{1}{4} R\, g_{ij}$ of the Ricci tensor R_{ij} derived from $g_{\mu\nu}$,

– the rescaled Weyl tensor $d^i_{jkl} = \Omega^{-1} C^i_{jkl}$ where C^i_{jkl} is the Weyl tensor of $g_{\mu\nu}$.

For these fields can be derived from (2.1) respectively (2.3) the following «regular conformal field equations» for the unknown

$$u = (e^\mu_k, \gamma^j_{ik}, \Omega, \Sigma_i, s, s_{ij}, d_{ijkl})$$

$$e^\mu_{k,\nu} e^\nu_j - e^\mu_{j,\nu}\, e^\nu_k = (\gamma^i_{jk} - \gamma^i_{kj})\, e^\mu_i \tag{2.6}$$

$$\gamma^i_{lj,\mu} e^\mu_k - \gamma^i_{kj,\mu} e^\mu_l + \gamma^i_{km} \gamma^m_{lj} - \gamma^i_{lm} \gamma^m_{kj} - \gamma^i_{mj} (\gamma^m_{kl} - \gamma^m_{lk}) = \\ = \Omega d^i_{jkl} + 2(g^i_{[k} s_{l]j} - g_{j[k} s_{l]}{}^i) + \frac{1}{6} R\, g^i_{[k} g_{l]j} \tag{2.7}$$

$$\nabla_i \Omega = \Sigma_i \tag{2.8}$$

$$\nabla_i \Sigma_k = -\Omega\, s_{ik} + s\, g_{ik} \tag{2.9}$$

$$\nabla_i s = -\Sigma^j s_{ji} - \frac{1}{12} R\, \Sigma_i - \frac{1}{24} \Omega \nabla_i R \tag{2.10}$$

$$\nabla_k s_{lj} - \nabla_l s_{kj} = \Sigma_i d^i_{jkl} - \frac{1}{12} g_{j[l} \nabla_{k]} R \tag{2.11}$$

$$\nabla_i d^i_{jkl} = 0. \tag{2.12}$$

These equations are obtained in the following way. Equ. (2.6) is the first structure equation with the condition that the torsion vanish, while equ. (2.7) is the second structure equation where on the right the curvature tensor is expressed in terms of its irreducible parts. Equ. (2.9) is the trace-free part of (2.3). Equ. (2.12) is obtained by rewriting the Bianchi-identity for the «physical Weyl-tensor», $\tilde{\nabla}_\mu \tilde{C}^\mu_{\lambda\nu\rho} = 0$, which is implied by (2.1), in terms of the non-physical quantities. Then equ. (2.11) is derived from the Bianchi identity for the non-physical curvature tensor by using (2.12). Finally, equ. (2.10) is an integrability condition derived form (2.9) and the other equations [7].

There are several aspects of the regular conformal field equations which deserve further discussion.

Since the unknown u does not comprise the Ricci scalar R the system appears to be underdetermined. However, the relation (2.2) and consequently equations

(2.6) - (2.12) are invariant under rescalings of the type

$$(2.13) \qquad (\Omega, g) \to (\hat{\Omega}, \hat{g}) = (\theta\Omega, \theta^2 g)$$

with a positive function θ, if all quantities given by u and also R are transformed appropriately (see (3.16)). This introduces, besides the possibility to choose coordinates and frame fields arbitrarily, an additional «gauge-freedom». It allows to determine near a suitable initial surface the conformal factor in such a way that the Ricci scalar acquires there any preassigned functional dependance $R = R(x^\mu)$ on the coordinates x^μ [9]. Thus the Ricci scalar R may in (2.6) - (2.12) be considered as an arbitrarily given function on $\mathbb{R}^4$ and the system (2.6) - (2.12) as an overdetermined system of partial differential equations for the unknown u.

Let now u be a smooth solution of (2.6) - (2.12) (satisfying $\det(e_k^\mu) \neq 0$) on a connected manifold M and denote by $\widetilde{M}$ the open submanifold $\{x \in M/\Omega(x) > 0\}$ of M. On $\widetilde{M}$ can be defined from u a metric $\widetilde{g}_{\mu\nu}$ be requiring $\widetilde{g}_{\mu\nu}(\Omega^{-2} g^{ik} e_i{}^\nu e_k^\lambda) = \delta_\mu^\lambda$. In the regular conformal field equations the cosmological constant does not appear since the trace of (2.3) has not been used in the derivation of (2.6) - (2.12). Therefore one may wonder, whether the metric $g_{\mu\nu}$ will be a solution of Einstein's equations (2.1). By taking the differential of the quantity

$$(2.14) \qquad \Lambda' = 6\Omega s - 3\Sigma_j \Sigma^j + \frac{1}{4}\,\Omega^2 R$$

and using that u is given by a solution of (2.6) - (2.12) one finds that $\Lambda' = \text{const}$ on each connected component of M. Identifying the cosmological constant Λ with Λ', equation (2.14) is just the trace of (2.3). Therefore one arrives at the fact: *Any solution of Einstein's equations (2.1) implies a solution of the regular conformal field equations (2.6) - (2.12). Any solution of the regular conformal field equations on a manifold M provides on the submanifold* $\widetilde{M} = \{x \in M/\Omega(x) > 0\}$ *a solution of Einstein's equations, where the cosmological constant is obtained as a constant of integration which may be fixed on a suitable initial surface. The regular conformal field equations generalize Einstein's equations since they are defined and regular for all values of the conformal factor, in particular at points where it vanishes.*

It is of course the latter property which motivated the introduction of the complicated system (2.6) - (2.12) as well as its name. Though it is already surprising that Einstein's equations allow such a generalization, the system (2.6) - (2.12) can only considered us useful if it allows to draw conclusions about it manifold of solutions. For this it is decisive that one can show: *The regular conformal field equations imply symmetric hyperbolic systems of propagation equations.*

There are various ways of extracting such propagation equations from (2.6) -

(2.12), depending on the choice of coordinates, frame field and conformal factor. This has been discussed in [7, 8] and in all possible generality in [9]. Initial value problems for equs. (2.6) - (2.12) may thus be reduced to initial value problems for symmetric hyperbolic systems. These systems have been studied extensively and very general existence results are available [6, 11, 16]. It may be mentioned finally, that it is possible to generalize the derivation given above, e.g. to obtain for the coupled Einstein-Yang-Mills equations regular conformal field equations which also imply symmetric hyperbolic propagation equations.

The conformal structure of Einstein's equations exhibited by the properties of the regular conformal field equations exactly match with the characterization of the asymptotic behaviour of gravitational fields in terms of conditions on their global conformal structure [18]. The space-times which are of interest in the following are described in the

DEFINITION (2.1). A connected, time- and space-orientable strongly causal space-time $(\tilde{M}, \tilde{g})$ *will be called asymptotically simple in the past*, if there exists a manifold M with boundary $\mathscr{I}^-(\subset M)$, a smooth Lorentz metric g and *a* function Ω on M, and an embedding $\phi : \tilde{M} \to M$, by means of which $\tilde{M}$ is identified with its image $\phi(\tilde{M}) = M \setminus \mathscr{I}^-$ such that

i) $\Omega > 0$ on $\tilde{M}$, $\Omega = 0$ on $\mathscr{I}^-$, $d\Omega \neq 0$ on $\mathscr{I}^-$,

ii) $g = \Omega^2 \tilde{g}$ on $\tilde{M}$,

iii) every null geodesic in $(\tilde{M}, \tilde{g})$ has (only) a past endpoint on $\mathscr{I}^-$.

The space-time $(\tilde{M}, \tilde{g})$ will be called *asymptotically simple and de Sitter in the past* if it satisfies in addition to (i) - (ii)

iv) on $\tilde{M}$ Einstein's equations Ric $[\tilde{g}] = \Lambda \tilde{g}$ hold with a cosmological constant $\Lambda > 0$.

These requirements have, among others, the following consequences [18]. All null geodesics of $(\tilde{M}, \tilde{g})$, which as point sets coincide with null geodesics in (M, g), are past complete. The gradient of Ω is timelike on $\mathscr{I}^-$ (see equ. (2.14)). Thus the surface $\mathscr{I}^-$, called «past conformal infinity» in the following, is a space-like hypersurface, which because of condition (iii) is in fact a Cauchy surface for the conformal extension (M, g) of $(\tilde{M}, \tilde{g})$ [12]. The physical Weyl tensor $\tilde{C}^{\mu}{}_{\nu\lambda\rho}$ obtained from $\tilde{g}_{\mu\nu}$, which coincides on $\tilde{M}$ with the Weyl tensor $C^{\mu}{}_{\nu\lambda\rho}$ for $g_{\mu\nu}$ goes to zero along any curve in M which approaches $\mathscr{I}^-$. (See equ. (2.12)). Because of this property it may be assumed that the rescaled Weyl tensor $\Omega^{-1}\tilde{C}^{\mu}{}_{\nu\lambda\rho}$ extends to a smooth tensorfield on M.

In a similar way as above one can define space-times which are asymptotically simple and de Sitter in the future with future conformal infinity $\mathscr{I}^+$ or which satisfy these conditions in the past as well as in the future. The prototype of all these cases in de Sitter space. It can as a whole be conformally embedded into the Einstein cosmos which is given by the manifold $\mathbb{R} \times S^3$ with the line element

$$ds^2 = -\mathrm{d}t^2 + \mathrm{d}\omega^2 \tag{2.15}$$

with $t \in \mathbb{R}$ and $\mathrm{d}\omega^2$ the standard line element on S^3. If one defines the function $\tau(t)$ by

$$\sinh \tau = tg\, t, \qquad -\frac{\pi}{2} < t < \frac{\pi}{2}$$

and the conformal factor by

$$\Omega = \cos t, \qquad -\frac{\pi}{2} \leqslant t \leqslant \frac{\pi}{2}$$

then one has on $\widetilde{M} = \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\times S^3$ in terms of the coordinate τ the de Sitter line element given by

$$\mathrm{d}\tilde{s}^2 = \Omega^{-2}\mathrm{d}s^2 = -\mathrm{d}\tau^2 + \cosh^2\tau\, \mathrm{d}\omega^2. \tag{2.16}$$

Thus the conformally extended space-time (M, g) is given by the closure of $\widetilde{M}$ in the Einstein cosmos together with the line element (2.15). The surfaces $\mathscr{I}^- = \left\{t = -\frac{\pi}{2}\right\}$ resp. $\mathscr{I}^+ = \left\{t = \frac{\pi}{2}\right\}$ represent past resp. future conformal infinity.

The purpose of condition (iii) is to make sure that nothing of $\mathscr{I}^-$ is left out in the construction of the conformally extended space-time. In the case that the manifold $\widetilde{M}$ has a compact Cauchy surface S (which appears to be a natural assumption for a solution of equ. (2.1) with $\Lambda > 0$, since there seem to be no natural conditions on the behaviour of the field «at spatial infinity») such that $\widetilde{M}$ is diffeomorphic to $\mathbb{R} \times S$, condition (iii) allows to conclude that $\mathscr{I}^-$ is diffeomorphic to S and the conformally extended space-time is diffeomorphic to $[0, \infty[\times S$. However there are space-time, called «*weakly asymptotically simple and de Sitter*» [19], for which the conditions (i, ii, iv) can be satisfied but (iii) does not hold because some null geodesics run into singualrities. Thus one can only attach certain «pieces of a smooth conformal infinity» to the original space-time. An example for this is provided by the Schwarzschild - de Sitter space-time and its analytic extensions which have been discussed in [13].

3. EXISTENCE AND CHARACTERIZATION OF SPACE-TIMES WHICH ARE ASYMPTOTICALLY SIMPLE AND DE SITTER IN THE PAST OR IN THE FUTURE

In the following will be investigated the set of space-times satisfying the conditions (i) - (iv) of definition (2.1) by analysing a Cauchy problem for the regular conformal field equations (2.6) - (2.12), where data are given on a space-like hypersurface S in the conformally extended space-time M. For this purpose suitable initial data sets, i.e. solutions of the constraint equations which are implied on space-like hypersurfaces by the regular conformal field equations, have to be determined.

Assume that the conformal factor Ω has been fixed somehow. Let the coordinates x^0, x^α and the frame vector fields e_0, e_a (in this chapter indices $\alpha, \beta, \ldots,$ $a, b, \ldots$ will take values 1, 2, 3 and the summation convention will be assumed for these indices) satisfying (2.4) be such that $S = \{x^0 = 0\}$, e_0 is the future directed unit normal to S so that the x^α provide coordinates on S and e_a an orthonormal frame for the interior metric implied on S. To extend the frame and the coordinates into a neighbourhood of S it is assumed furthermore that the frame e_k is parallely propagated in the direction of e_0, x^0 is a parameter of the integral curves of e_0 and the x^α are dragged along with e_0:

$$\nabla_0 e_k = 0, \quad x^\mu{}_{,\nu} e_0^\nu = \delta_0^\mu \qquad \text{on } M \text{ near } S.$$

The coordinates x^μ are thus Gauss coordinates based on S. One finds

$$e_0 = \frac{\partial}{\partial x^0}, \quad e_a = e_a^\alpha \frac{\partial}{\partial x^\alpha}$$

$$\gamma^j_{0k} = 0; \quad \gamma^0_{ab} = \gamma^b_{a0} = \chi_{ab} \quad \text{is the second fundamental form on } S.$$

Let D denote the covariant Levi-Civita derivative operator implied on S by the interior metric induced on S by $g_{\mu\nu}$. Then one has

$$D_a e_b = D_{e_a} e_b = \gamma^c_{ab}\, e_c,$$

with the γ's given by (2.5) and the torsion ${}^3t^b_{ac}$ for D vanishes. The curvature tensor, the Ricci tensor, and the Ricci scalar derived from D will be denoted by ${}^3r^a_{bcd'}$, ${}^3r_{ab}$, ${}^3r$ respectively. The Bach tensor on S is then given by

$${}^3b_{cab} = D_{[a}\,{}^3r_{b]c} - \frac{1}{4} D_{[a}\,{}^3r\, g_{b]c}.$$

To obtain simple expressions for the constraint equations it is convenient to introduce the notation

$$\Sigma = \Sigma_{0'}, \quad s_a = s_{a0'}, \quad d_{ab} = d_{a0b0'}, \quad d_{abc} = d_{a0bc}.$$

These fields on S have the algebraic properties

$$s^a_a = s_{00'}, \quad d_{ab} = d_{ba'}, \quad d^a_a = 0, \quad d_{abc} = -d_{acb'}, \quad d_{[abc]} = 0$$

$$d^a_{ac} = 0, \quad d_{eabc} = 2(g_{e[b}d_{c]a} + g_{a[c}d_{b]e})$$

and contain all the information on s_{ik} and d_{ijkl}.

The constraint equations implied on S by equs. (2.6) - (2.12) are given now by

(3.1) $$^3r^e_{abc} = -2\chi^e_{[b}\chi_{c]a} + \Omega\, d^e_{abc} + \frac{1}{6} R\, g^e_{[b} g_{c]a} + 2(g^e_{[b}s_{c]a} - g_{a[b}s_{c]}{}^e)$$

(3.2) $$D_b\chi_{ca} - D_c\chi_{ba} = \Omega\, d_{abc} + 2g_{a[b}s_{c]}$$

(3.3) $$D_a\Omega = \Sigma_a$$

(3.4) $$D_a\Sigma_b = \Sigma\chi_{ab} - \Omega\, s_{ab} + s\, g_{ab}$$

(3.5) $$D_a\Sigma = \Sigma^c\chi_{ca} - \Omega\, s_a$$

(3.6) $$D_a s = \Sigma\, s_a - \Sigma^c s_{ca} - \frac{1}{12} R\,\Sigma_a - \frac{1}{24}\Omega\, D_a R$$

(3.7) $$D_a s_b - D_b s_a = 2\chi_{[a}{}^c s_{b]c} + \Sigma^c d_{cab}$$

(3.8) $$D_a s_{bc} - D_b s_{ac} = 2\chi_{c[a}s_{b]} + \Sigma d_{cab} + \Sigma^e d_{ecab} + \frac{1}{12} g_{c[a}D_{b]}R$$

(3.9) $$D_a d^a_{bc} = 2d_{e[b}\chi^e_{c]}$$

(3.10) $$D_a d^a_b = \chi^{ef}d_{ebf}.$$

Furthermore (2.14) gives the equation

(3.11) $$\Lambda = 6\Omega s + 3\Sigma^2 - 3\Sigma_a\Sigma^a + \frac{1}{4}\Omega^2 R$$

which is implied on S also by (3.1) - (3.10).

As the initial surface is now chosen the surface $S = \mathscr{I}^-$. Then the constraint equations (3.1) - (3.11) simplify enormously since Ω vanishes. An initial data set may then the obtained in the following way:

LEMMA (3.1). *Let (S, h) be a connected orientable 3-dimensional Riemannian space with manifold S and metric h tensor on S. Let R and τ be real-valued functions on S and Λ a positive number. Furthermore let $d_{\alpha\beta}$ be a symmetric tracefree covariant tensor field on S datisfying*

$$D_\alpha d^\alpha_\beta = 0 \tag{3.12}$$

where D denotes the covariant Levi-Civita derivative operator defined by h. From these fields an initial data set

$$u_0 = (e^\mu_k, \gamma^j_{ik}, \Omega, \Sigma_i, s, s_{ij}, d_{ijkl}) \quad on\ S$$

for the regular conformal field equations (2.6)-(2.12) which satisfies (2.14) with $\Lambda = \Lambda'$ on S is determined as follows.

Let $c_a = c^\alpha_a \dfrac{\partial}{\partial x^\alpha_\alpha}$ be a (local) frame given with respect to some (local) co-ordinate system x on S such that

$$h(c_a, c_b) = g_{ab} = \mathrm{diag}\,(1, 1, 1)$$

and let Γ^b_{ac} be the connection coefficients satisfying

$$D_a c_b = D_{c_a} c_b = \Gamma^d_{ab} c_d .$$

Furthermore denote by ${}^3R_{ab}$, ${}^3R$, ${}^3B_{cab}$ the Ricci-tensor, the Ricci scalar and the Bach tensor obtained from h. Then u_0 is given by

$$e^\mu_0 = \delta^\mu_0, \quad e^0_a = 0, \quad e^\alpha_a = c^\alpha_a, \quad \gamma^b_{ac} = \Gamma^b_{ac} \tag{3.13}$$

$$\begin{cases} \Omega = 0, \ \Sigma_a = 0, \ s = \tau\left(\dfrac{1}{3}\Lambda\right)^{1/2} \\ \Sigma = \left(\dfrac{1}{3}\Lambda\right)^{1/2}, \ \chi_{ab} = -\tau g_{ab}, \ s_a = D_a \tau \end{cases} \tag{3.14}$$

$$\begin{cases} s_{ab} = {}^3R_{ab} - g_{ab}\left(\dfrac{1}{4}\,{}^3R + \dfrac{1}{24}R - \dfrac{1}{2}\tau^2\right) \\ d_{a0bc} = \left(\dfrac{1}{12}\Lambda\right)^{-1/2} {}^3B_{abc} \\ d_{a0b0} = d_{\alpha\beta} c^\alpha_a c^\beta_b . \end{cases} \tag{3.15}$$

Proof. Equs. (3.13) follow from the preceding choice of the coordinates and the frame. Assuming that the cosmological constant is given by Λ, that $\Omega = 0$ on S

and setting $s = \tau\left(\frac{1}{3}\Lambda\right)^{1/2}$, equs. (3.14) are a consequence of and imply equs. (3.3), (3.11), (3.4), and (3.6). The equs. (3.2), (3.5), (3.7) are then satisfied identically. Using the fact that in a 3-dimensional space the curvature tensor is determined by the Ricci tensor via the idenity

$$^3r_{abcd} = 2(g_{a[c}\,{}^3r_{d]b} + g_{b[d}\,{}^3r_{c]a}) + {}^3r\, g_{a[d}g_{c]b},$$

the remaining constraints can be written equivalently

$$(3.1') \qquad {}^3r_{ab} = s_{ab} + g_{ab}\left(s^c_c + \frac{1}{6}R - 2\tau^2\right)$$

$$(3.8') \qquad d_{cab} = 2\Sigma^{-1}\,{}^3b_{cab}$$

$$(3.9') \qquad D_c d^c_{ab} = 0$$

$$(3.10') \qquad D_c d^c_a = 0.$$

Hence (3.1′) and (3.8′) may be solved by using them as defining equations for s_{ab} and d_{cab}. Then (3.9′) is just the differential identity satisfied by the Bach tensor. Finally, (3.10′) is equivalent to (3.12). ■

Remarks

i) The functions (3.13) depend of course on the specific choice of the coordinates and the frame. The present choice gives immediately the appropriate data for the symmetric hyperbolic equations as derived from (2.6) - (2.12) in [8] for the case of a conformal factor such that $R = 0$ near the initial surface. Important here is the way the metric h with $h^{\alpha\beta} = g^{ab}\, e^\alpha_a\, e^\beta_b$ and the tensorial quantities (3.14), (3.15) are determined, which is independent of any choice of gauge.

ii) It is remarkable that the data on past conformal infinity for any space--time which is (weakly) asymptotically simple and de Sitter may be obtained as described in Lemma (3.1). While in the standard Cauchy problem for Einstein's equations the Hamiltonian constraint leads to an elliptic equation, an analogous equation does not occur here. The only differential relation is equation (3.12), which may be considered as an analogue of the momentum constraint.

iii) If the coupled Einstein-Yang-Mills equations are transformed into a regular conformal system, the initial data u_0 on $\mathscr{I}^-$ for the geometrical quantities can be, with the exception of $d_{\alpha\beta}$, determined in the same way as above. If then a solution of the constraints implied by the Yang-Mills equations on $\mathscr{I}^-$ has been found, an equation for $d_{\alpha\beta}$ has to be solved which is of the form (3.12) with a right member which is a quadratic expression of the Yang-Mills fields.

iv) At first sight it may appear that there is too much freedom here to specify initial data for the gravitational field [5]. This, however, is not the case. Under the rescaling (2.13) the Ricci scalar $\hat{R} = \hat{R}[g]$ is transformed into the Ricci scalar $R = R[\theta^2 g]$ according to

$$(3.16) \qquad \nabla_\mu \nabla^\mu \theta = \frac{1}{6}(\theta R - \theta^3 \hat{R}).$$

As mentioned in the beginning this allows to choose the Ricci scalar $\hat{R}$ freely near the initial surface. When equ. (3.16) is then solved for the function θ there is still the freedom to specify a positive function θ and an arbitrary function, the normal derivative $\nabla_0 \theta$, on the initial surface. On the surface $\mathscr{I}^-$, where $\Omega = 0$, the quantity $\tau = \Sigma^{-1} s$ transforms under (2.13) according to

$$(3.17) \qquad \theta^2 \tau = \theta \hat{\tau} - \nabla_0 \theta .$$

This shows that for arbitrarily given positive θ the function $\nabla_0\theta$ can be chosen such that $\hat{\tau}$ takes any preassigned value on the initial surface. In particular one may choose $\tau \equiv 0$, $R \equiv 0$ in (3.14), (3.15). There remains the freedom to choose θ on S. If the fields $h_{\alpha\beta}$, $d_{\alpha\beta}$ on S are rescaled in the form

$$(3.18) \qquad h_{\alpha\beta} \to \hat{h}_{\alpha\beta} = \theta^2 h_{\alpha\beta}, \quad d_{\alpha\beta} \to \hat{d}_{\alpha\beta} = \theta^{-1} d_{\alpha\beta}$$

equation (3.12) will again be satisfied by the transformed fields. From $\hat{h}$ and $\hat{d}$ may be derived an initial data set $\hat{u}_0$ as described in Lemma (3.1) with $\hat{R}$, $\hat{\tau}$ given arbitrarily and with the same constant Λ. The initial data sets u_0, $\hat{u}_0$ will then be called conformally equivalent. As stated in Theorem (3.2) they will determine isometric solutions $\tilde{g}_{\mu\nu}$ of Einstein's equations (2.1).

Let (S, u_0) be an initial data set as determined in Lemma (3.1) where the manifold S is now assumed to be compact. A manifold M with boundary $\mathscr{I}^-$ together with a collection $u = (e^\mu_k, \ldots, d_{ijkl})$ of fields on M will be called a solution of the initial value problem for the regular conformal field equations with initial data set (S, u_0), if M is diffeomorphic to $S \times [0, T[$ for some $T > 0$, u is a solution of equations (2.6) - (2.12) on M, the function Ω is positive on M except on the boundary $\mathscr{I}^-$, and there exists an embedding of S into M by which S is identified with the boundary $\mathscr{I}^-$ of M in such a way that the fields given by u_0 coincide (possibly after a rotation of the frame) with the fields implied by u on $\mathscr{I}^-$.

In this formulation it has been assumed for convenience, that the frame c_a introduced on S in Lemma (3.1) and consequently the frame e_k is given globally on S resp. M. The definition has to be generalized in an obvious way if more than one frame is involved in the construction of the initial data set.

If k is an integer $\geqslant 0$, let $H^k(S)$ resp. $H^k(M)$ denote L^2-type Sobolev spaces on S resp. on M. Here $H^k(S)$ is defined with respect to the bilinear product of tensor fields of the same type and the measure implied on S by some fixed Riemannian metric and $H^k(M)$ is defined in a similar way by using the product metric implied on $S \times [0, T[$ by the given metric on S and the standard metric on $\mathbb{R}$.

THEOREM (3.2). *Suppose $k \geqslant 4$ and (S, u_0) is an initial data set as determined in Lemma (3.1) with a smooth and compact manifold S and with fields provided by u_0 on S, which are of class $H^k(S)$. Then there exists a unique (up to questions of extensibility) solution (M, u) of the initial value problem for the regular conformal field equations (2.6) - (2.12) with initial data (S, u_0), such that the fields supplied by u are of class $H^k(M)$.*

In particular the conformal factor Ω and the metric $g_{\mu\nu}$ provided by u are of class $H^k(M)$. The space-time with manifold $\tilde{M} = M \setminus \mathscr{I}^-$ and metric $\tilde{g}_{\mu\nu} = \Omega^{-2} g_{\mu\nu}$ is a solution of Einstein's field equations

$$\mathrm{Ric}\,[\tilde{g}] = \Lambda\, \tilde{g} \quad \textit{(with } \Lambda > 0 \textit{ as given by } u_0\textit{)}$$

which is past asymptotically simple.

The isometry class of the space-time $(\tilde{M}, \tilde{g}_{\mu\nu})$ determined from (S, u_0) is not changed if the initial data set (S, u_0) is replaced by a conformally equivalent initial data set.

Remarks

i) As seen from Lemma (3.1) for the fields given by u_0 to be of class H^k it is sufficient that the metric h is of class H^{k+3}, the field d and the function R are of class H^k and the function τ is of class H^{k+1}. No attempt has been made here to formulate the strongest possible results regarding differentiability (see [6]).

ii) The assumption that the manifold S be compact has been made here to simplify the statement of the theorem. For more general situations, corresponding to space-times which may form a part of a weakly asymptotically simple space-time, a similar existence theorem can be formulated which involves local Sobolev spaces.

iii) It is a remarkable fact that all space-times satisfying the conditions of definition (2.1) can be obtained in a similar way as described in Lemma (3.1) and Theorem (3.2) and that the degree of freedom to specify initial data at past (or future) conformal infinity is essentially the same as in the case of the standard Cauchy problem for Einstein's field equations.

Proof of Theorem (3.1). Using the standard method of localization for symmetric hyperbolic systems, the existence part follows by taking a finite covering of S by coordinate patches which carry an orthonormal frame for the metric h and by constructing local solutions. Applying a general theorem of Kato [16] on the Cauchy problem for symmetric hyperbolic systems to the symmetric hyperbolic propagation equations implied by equations (2.6) - (2.12) the existence of local solutions can be established. This has been worked out in detail in [8] where it also has been shown that a solution of the symmetric hyperbolic propagation equations which solves the constraint equations on the initial surface is in fact a solution of (2.6) - (2.12). That the isometry class of the physical space-time $(\widetilde{M}, \widetilde{g})$ does not depend on which element of the conformal equivalence class of initial data sets has been chosen for its construction has been shown in [9]. ■

4. WEAKLY ASYMPTOTICALLY SIMPLE AND DE SITTER SPACE-TIMES WITH SYMMETRIES

In their article [13] Gibbons and Hawking discuss among other things the situation of an observer, who is travelling in a space-time which is weakly asymptotically simple and de Sitter along a time-like curve λ which has endpoint p on $\mathscr{I}^+$. Near p the intersection of $I^-(\lambda)$, the chronological past of λ [15], with a sufficiently extended space-like hypersurface will have compact support. Consequently, they argue, there will only a finite amount of energy available to be radiated through the cosmological event horizon $\dot{I}^-(\lambda)$ of the observer and therefore his space-time will eventually approach a stationary state. The following result suggests that the space-time of the observer will then in effect become conformally flat.

THEOREM (4.1). *Assume that on a weakly asymptotically simple and de Sitter space-time there exists a Killing vector field K which has a time-like integral curve κ that enters each neighbourhood of a certain point $p \in \mathscr{I}^+$. Then a neighbourhood of p in $J^-(p)$, the causal past of p in the conformally extended space--time, will be conformally flat.*

Proof. First a few properties of Killing vector fields on space-times satisfying the requirements of definition (2.1) will be discussed.

Let K be a Killing vector field on $\widetilde{M}$ for $\widetilde{g}$. Then K is a conformal Killing vector field on $\widetilde{M}$ for $g = \Omega^2 \widetilde{g}$, i.e. $K_\mu = g_{\mu\nu} K^\nu$ satisfies (in the notation of chapter 2) the conformal Killing equations

$$\nabla_i K_j + \nabla_j K_i - \frac{1}{2} \nabla_l K^l g_{ij} \equiv U_{ij} = 0$$

on $\tilde{M}$. The field K has then a smooth extension to a conformal Killing vector field on (M, g) which will again be denoted by K. In fact the equations $U_{0j} = 0$ imply a linear, homogeneous symmetric hyperbolic system of propagation equations for K_j which yields the smooth extension of K and the remaining equations $U_{ab} = 0$ will be satisfied by continuity. Incidentally, this shows that K will vanish everywhere on $D^-(\mathscr{I}^+)$ if it vanishes on $\mathscr{I}^+$ everywhere.

For the following it is important that the flow of the field K maps null geodesics onto nullgeodesics and that $g(K, \eta') = \text{const}$ along any null geodesic η with tangent vector η'.

Since K comes from a Killing field on $(\tilde{M}, \tilde{g})$, its flow must map (future) endpoints of null geodesics in $(\tilde{M}, \tilde{g})$, i.e. points of $\mathscr{I}^+$ onto such points. Thus K is tangent to whence space-like on $\mathscr{I}^+$. If the integral curve $\kappa = \kappa(s)$ of K has endpoint $p \in \mathscr{I}^-$, then $K(p) = 0$. However, p is an isolated critical point of the field K, since along any future directed null geodesic η passing through a point $\kappa(s)$ one has $g(K, \eta') = \text{const} < 0$, as at $\kappa(s)$ the field K is time-like.

The following argument will hold in a suitably chosen (non-empty) neighbourhood U of p. The set $N_p = I^-(\kappa, U) \setminus \{p\}$ is a smooth null hypersurface swept out by the past directed null geodesics through p. Since p is a fixed point of the flow of K it follows that K is tangent to N_p. The convergence $\tilde{\rho}$ and the shear $\tilde{\sigma}$ (using now Newman-Penrose notation [17] with respect to a pseudo orthonormal frame $\tilde{l}, \tilde{n}, \tilde{m}, \bar{\tilde{m}}$ for $\tilde{g}$ such that $\tilde{l}$ is tangent to the null generators of N_p and the frame is parallely transported in the direction of $\tilde{l}$) of N_p must vanish (see also [4, 14] for the argument). It is clear that one has $\tilde{\rho} \leqslant 0$ on N_p, since otherwise because of the equation

$$\tilde{D}\,\tilde{\rho} = \tilde{\rho}^2 + \tilde{\sigma}\bar{\tilde{\sigma}} \tag{4.1}$$

a caustic would form. Consider now the spherical surface Σ_s which is the intersection of N_p with the set of future directed null geodesics emanating from a point $\kappa(s)$ near p. Since K is timelike on κ, the surface $\Sigma_{s'}$ will be in the future of Σ_s if $s' > s$. Furthermore Σ_s is mapped isometrically onto $\Sigma_{s'}$ by the flow of K. From this one finds

$$0 = \frac{\mathrm{d}}{\mathrm{d}s} \int_{\Sigma_s} \mathrm{d}\tilde{A} = \int_{\Sigma_s} f \tilde{\rho}\, \mathrm{d}\tilde{A}$$

where $\mathrm{d}\tilde{A}$ is the surface element and f a function which is negative everywhere on Σ_s. This implies that $\tilde{\rho}$ vanishes on Σ_s for all large s and hence, by suitable choice of U, on N_p. From equation (4.1) then follows $\tilde{\sigma} \equiv 0$ on N_p and from $D\tilde{\sigma} = 2\tilde{\rho}\,\tilde{\sigma} + \tilde{\Psi}_0$ one obtains for the Weyl tensor term

(4.2) $$\tilde{\Psi}_0 \equiv 0 \quad \text{on} \quad N_p.$$

Together with (4.2) equation (2.12) implies that $J^-(p)$ is conformally flat near p. In the spin frame formalism (now with respect to the non-physical metric g) equation (2.12) reads

(4.3) $$\nabla^a_{a'} \varphi_{abcd} = 0$$

where φ_{abcd} represents the rescaled Weyl spinor. To show that φ_{abcd} must vanish on N_p assume that (4.3) is expressed in a spin frame $(\iota_a)_{a=0,1}$ such that the vector corresponding to $\iota_0 \bar{\iota}_{0'}$ is tangent to the null generators of N_p. Once φ_{0000} is then known on N_p, the equation

$$\nabla_{00'} \varphi_{1000} - \nabla_{10'} \varphi_{0000} = -\nabla^a_{0'} \varphi_{a000} = 0$$

obtained from (4.3) is an ordinary differential equation (singular at p) along the null generators of N_p. For given φ_{1000} one obtains a similar equation for φ_{1100} and so on. It has been shown in detail in [10] that this hierarchy of ordinary differential equations allows to determine φ_{abcd} uniquely on N_p once φ_{0000} is given there. The initial values for φ_{abcd} at p tollow from the reguirement that φ_{abcd} represent a smooth spinor field near p. However, equation (4.2) is equivalent to

$$\varphi_{0000} \equiv 0 \quad \text{on} \quad N_p$$

whence (4.3) entails

(4.4) $$\varphi_{abcd} \equiv 0 \quad \text{on} \quad N_p$$

which then must hold for any choice of spinframe. Assume now that (4.3) has been expressed with respect to a spin frame which is smooth near p. Then (4.3) implies the linear homogeneous symmetric hyperbolic system

(4.5) $$\begin{aligned} -\nabla^f_{0'} \varphi_{111f} &= 0 \\ -\nabla^f_{0'} \varphi_{ab0f} + \nabla^f_{1'} \varphi_{ab1f} &= 0 \\ \nabla^f_{1'} \varphi_{000f} &= 0. \end{aligned}$$

By a standard uniqueness argument for symmetric hyperbolic systems, which has been discussed in [8] for the system (4.5) in the notation of chapter 2, it follows then that (4.4) and (4.5) imply

$$\varphi_{abcd} \equiv 0 \quad \text{on} \quad D^-(N_p \cup p).$$ ■

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